ВТОРАЯ МЕЖДУНАРОДНАЯ МАТЕМАТИЧЕСКАЯ ОЛИМПИАДА
БЛАГОВЕЩЕНСК – РОССИЯ, 19 марта 2022

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第二届国际数学奥林匹克竞赛

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Problem statements and solutions

***Problem 1* (9 points)**

Determine the number of zeros of the function

$$f(x) = 2e^{2-x^2}(x^6 - 3x^4 + 5x^2 - 1) - 2e - 5, \quad x \in R.$$

Answer: function $f(x)$ has four (4) zeros.

Solution:

Function $f(x)$ is even, so it is enough to consider the interval $x \geq 0$. Here the number of zeros of the function $f(x)$ coincides with the number of zeros of the function

$$g(t) = 2e^{2-t}(t^3 - 3t^2 + 5t - 1) - 2e - 5.$$

Consider

$$g'(t) = 2e^{2-t}(-t^3 + 6t^2 - 11t + 6) = -2e^{2-t}(t-1)(t-2)(t-3).$$

We define the monotonicity intervals of the function $g(t)$:

$(0,1), (2,3)$ are increasing intervals; $(1,2), (3,+\infty)$ are decreasing intervals.

We check the values of the function $g(t)$ at the extremum points:

$$g(0) = -2e^2 - 2e - 5 < 0,$$
$$g(1) = 4e - 2e - 5 > 0,$$
$$g(2) = 10 - 2e - 5 < 0,$$
$$g(3) = 28e^{-1} - 2e - 5 < 0.$$

The last inequality follows from

$$28 - 2e^2 - 5e < 28 - 2(2{,}7)^2 - 5 \cdot 2{,}7 < 0.$$

It follows that the continuous function $g(t)$ has zeros at intervals $(0,1)$ and $(1,2)$. Accordingly, the continuous function $f(x)$ has only four (4) zeros нуля at intervals $\left(-\sqrt{2},\ -1\right),\ (-1,0),\ (0,1),\ \left(1,\sqrt{2}\right)$.

***Problem 2* (12 points)**

Find the limit

$$\lim_{x\to 0}\left(\lim_{n\to\infty}\frac{1}{x}(A^n - E)\right),$$

where

$$A = \begin{pmatrix} 1 & \dfrac{x}{n} \\ -\dfrac{x}{n} & 1 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Answer: $E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Solution:

Write

$$A = \frac{x}{n}E_1 + E$$

and note

$$E_1^2 = E_1 \cdot E_1 = -E,\ \ E_1^3 = -E_1,\ \ E_1^4 = E,\ \ E_1^5 = E_1, \dots$$

Then

$$A^n = \sum_{k=0}^{n} C_n^k E_1^k \left(\frac{x}{n}\right)^k = E \cdot \sum_{k=0}^{k\le\frac{n}{2}} (-1)^k C_n^{2k} \left(\frac{x}{n}\right)^{2k} + E_1 \cdot \sum_{k=0}^{k\le\frac{n}{2}} (-1)^k C_n^{2k+1} \left(\frac{x}{n}\right)^{2k+1};$$

Considering then when $n \to \infty$

$$u_k = C_n^k \left(\frac{x}{n}\right)^k, \quad \left|\frac{u_{k+1}}{u_k}\right| = \frac{C_n^{k+1}}{C_n^k}\frac{|x|}{n} = \frac{(n-k)|x|}{(k+1)n} \to \frac{|x|}{k+1},$$

all rows converge as Leibniz rows if $|x| < 1$.

So

$$\frac{1}{x}(A^n - E) = E_1 + \left(\frac{x}{n}\right)^2 E_1 \cdot \sum_{k=1}^{k\le\frac{n}{2}} (-1)^k C_n^{2k+1} \left(\frac{x}{n}\right)^{2k-2} + \frac{x}{n} \cdot E$$

$$\cdot \sum_{k=1}^{k\le\frac{n}{2}} (-1)^k C_n^{2k+1} \left(\frac{x}{n}\right)^{2k-1} \to E_1$$

by $x \to 0$.

***Problem 3* (10 points)**

Find the sum

$$f(0)+f\left(\frac{1}{2022}\right)+f\left(\frac{2}{2022}\right)+f\left(\frac{3}{2022}\right)+\cdots+f\left(\frac{2021}{2022}\right)+f\left(\frac{2022}{2022}\right)$$

for given function

$$f(x)=\frac{a^{2x}}{a^{2x}+a}\ , \qquad a>0,\ \ x\in R.$$

$$f(0)+f\left(\frac{1}{2022}\right)+f\left(\frac{2}{2022}\right)+f\left(\frac{3}{2022}\right)+\cdots+f\left(\frac{2021}{2022}\right)+f\left(\frac{2022}{2022}\right).$$

Answer: $\frac{2023}{2}$

Solution:

Note

$$f(x)=\frac{a^{2x}}{a^{2x}+a}=1-\frac{a}{a^{2x}+a}=1-\frac{a^{2(1-x)}}{a^{2(1-x)}+a}=1-f(1-x),$$

so $f(x)+f(1-x)=1$.

Then we get

$$f(0)+f\left(\frac{1}{2022}\right)+f\left(\frac{2}{2022}\right)+f\left(\frac{3}{2022}\right)+\cdots+f\left(\frac{2021}{2022}\right)+f\left(\frac{2022}{2022}\right)=$$

$$\left[f(0)+f\left(\frac{2022}{2022}\right)\right]+\left[f\left(\frac{1}{2022}\right)+f\left(\frac{2021}{2022}\right)\right]+\left[f\left(\frac{2}{2022}\right)+f\left(\frac{2020}{2022}\right)\right]+\cdots$$

$$+\left[f\left(\frac{1009}{2022}\right)+f\left(\frac{1013}{2022}\right)\right]+\left[f\left(\frac{1010}{2022}\right)+f\left(\frac{1012}{2022}\right)\right]+f\left(\frac{1011}{2022}\right)=$$

$$=[f(0)+f(1-0)]+\left[f\left(\frac{1}{2022}\right)+f\left(1-\frac{1}{2022}\right)\right]+\left[f\left(\frac{2}{2022}\right)+f\left(1-\frac{2}{2022}\right)\right]$$

$$+\cdots$$

$$+\left[f\left(\frac{1009}{2022}\right)+f\left(1-\frac{1009}{2022}\right)\right]+\left[f\left(\frac{1010}{2022}\right)+f\left(1-\frac{1010}{2022}\right)\right]+f\left(\frac{1}{2}\right)=$$

$$=1011\cdot 1+\frac{a}{a+a}=1011+\frac{1}{2}=\frac{2023}{2}.$$

Problem 4 ***(9 points)***

Draw a line defined by the complex equation (t is a real parameter):

$$z\cdot\left(1+e^{-it}\right)^2=1.$$

Answer: $y^2=\frac{1}{4}-x.$

Solution:

$$z\left(1+e^{-it}\right)^2=1\ \Rightarrow\ z\left(\frac{e^{i\frac{t}{2}}+e^{-i\frac{t}{2}}}{e^{i\frac{t}{2}}}\right)^2=1\ \Rightarrow\ z=\frac{e^{it}}{4cos^2\frac{t}{2}}=\frac{cost+i\,sint}{4cos^2\frac{t}{2}}=$$

$$=\frac{1}{4}\left(\left(1-tg^2\frac{t}{2}\right)+2i*tg\frac{t}{2}\right).$$

Denote

$$\tau=tg\frac{t}{2}\,,\quad \tau\in(-\infty,+\infty)$$

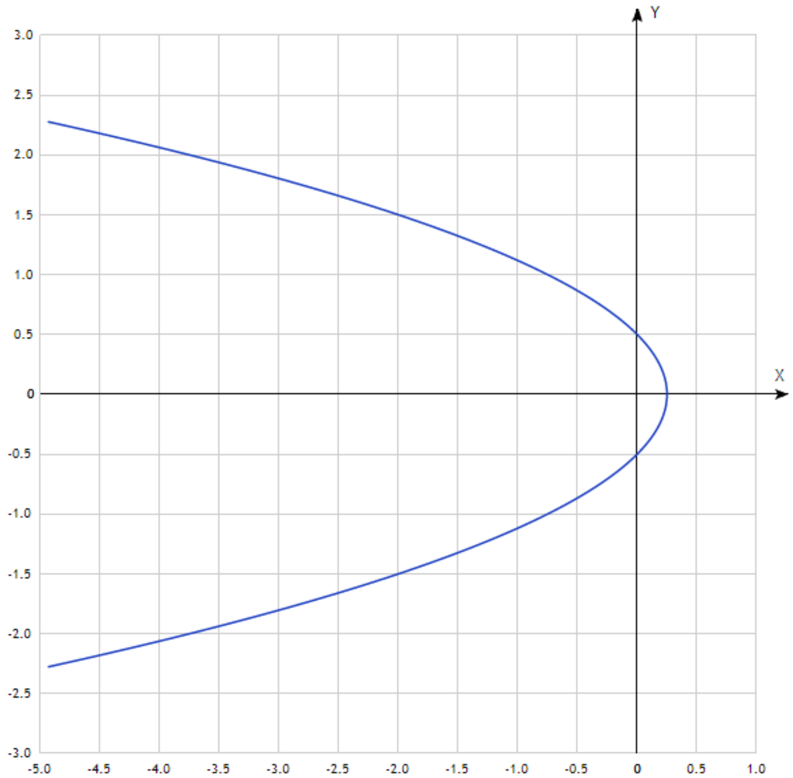
then

$$x=\frac{1}{4}\left(1-tg^2\frac{t}{2}\right)=\frac{1}{4}(1-\tau^2),\qquad y=\frac{1}{2}tg\frac{t}{2}=\frac{1}{2}\tau\qquad\Rightarrow\qquad x=\frac{1}{4}(1-4y^2),$$

or

$$y^2=\frac{1}{4}-x.$$

The last is the parabola equation with a vertex at a point $(1/4;0)$, branches are directed to the left.



***Problem 5* (9 points)**

Find the limit

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\int_1^n ln\left(1+\frac{1}{\sqrt{x}}\right)dx.$$

Answer: **2**

Solution:

We use the Lopital rule:

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\int_1^n ln\left(1+\frac{1}{\sqrt{x}}\right)dx = \lim_{t\to\infty}\frac{1}{\sqrt{t}}\int_1^t ln\left(1+\frac{1}{\sqrt{x}}\right)dx = \lim_{t\to\infty}\frac{ln\left(1+\frac{1}{\sqrt{t}}\right)}{\frac{1}{2\sqrt{t}}} = 2.$$

***Problem 6* (8 points)**

Find the sum of the series

$$\sum_{n=1}^{\infty}\frac{n}{2^n}.$$

Answer: **2**

Solution:

Obviously, this series converges. We use the independence oproperty of the convergent series sum from the permutation of its terms and the sum formula of an infinitely decreasing geometric progression

$$\sum_{n=1}^{\infty}\frac{n}{2^n} = \frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots$$

$$= \frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right)+\cdots =$$

$$= \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)+\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)+\left(\frac{1}{8}+\frac{1}{16}+\cdots\right)+\cdots =$$

$$= 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 2.$$

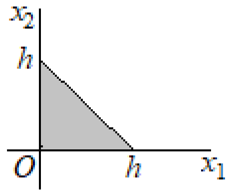
***Problem 7* (12 points)**

Find the volume of the m-dimensional pyramid T_m such that

$$x_1 \geq 0,\ x_2 \geq 0, \cdots, x_m \geq 0,\ x_1 +\ x_2 + \cdots + x_m \leq h.$$

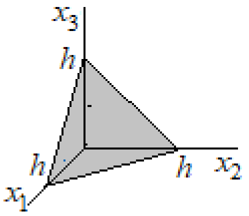
Answer: $\frac{h^m}{m!}$

Solution:



Consider $k = 2,\ V_2 = \iint_S dx_1 dx_2 = \int_0^h dx_1 \int_0^{h-x_1} dx_2 =$

$$= \int_0^h (h - x_1) dx_1 = -\frac{(h-x_1)^2}{2}\Big|_0^h = \frac{h^2}{2} = \frac{h^2}{2!},$$



$$k = 3,\ V_3 = \iiint\limits_V dx_1 dx_2 dx_3 = \int_0^h dx_1 \int_0^{h-x_1} dx_2 \int_0^{h-x_1-x_2} dx_3 =$$

$$= \int_0^h dx_1 \int_0^{h-x_1} (h - x_1 - x_2) dx_2 = -\int_0^h dx_1 \frac{(h-x_1-x_2)^2}{2}\Big|_0^{h-x_1} =$$

$$= \int_0^h \frac{(h-x_1)^2}{2} dx_1 = -\frac{(h-x_1)^3}{6}\Big|_0^h = \frac{h^3}{3!},$$

then by induction

$$k = m,\ V_m = \iiint\limits_V dx_1 dx_2 \cdots dx_m = \int_0^h dx_1 \int_0^{h-x_1} dx_2 \ldots \int_0^{h-x_1-x_2-\ldots-x_{m-1}} dx_m =$$

$$= \int_0^h dx_1 \int_0^{h-x_1} dx_2 \ldots \int_0^{h-x_1-x_2-\ldots-x_{m-2}} (h - x_1 - x_2 - \ldots - x_{m-1}) dx_{m-1} = \frac{h^m}{m!}.$$

***Problem 8* (12 points)**

Find real solutions of the differential equation

$$(y')^3 + \frac{2y'}{x^2} = 1 + \frac{2}{x^2} + \frac{6y}{x} + \frac{12y^2}{x^2} + \frac{8y^3}{x^3} + \frac{4y}{x^3}.$$

Answer: $y = Cx^2 - x.$

Solution:

Multiply everything by x^3:

$$(xy')^3 + 2xy' = x^3 + 2x + 6yx^2 + 12y^2x + 8y^3 + 4y.$$

Convert:

$$(xy')^3 + 2xy' = (x + 2y)^3 + 2(x + 2y).$$

Denote:

$$z = xy',\ t = x + 2y \ \Rightarrow\ z^3 + 2z = t^3 + 2t \ \Rightarrow\ (z - t)(z^2 + zt + t^2 + 2) = 0.$$

The real solution is answered by the case

$$z = t \ \Rightarrow\ xy' = x + 2y \ \Rightarrow\ y' = 1 + 2\frac{y}{x}.$$

We have a homogeneous differential equation of the 1st order, which is solved by replacing

$$y = tx,\ \ y' = t'x + t,$$

where $t = t(x)$ is a new function; next, we solve

$$t'x + t = 1 + 2t \ \Rightarrow\ t'x = 1 + t \Rightarrow\ \frac{dt}{1+t} = \frac{dx}{x} \Rightarrow\ ln|1 + t| = lnCx \Rightarrow\ 1 + t = Cx.$$

We return to the function we are looking for

$$y = Cx^2 - x.$$

***Problem 9* (9 points)**

Suppose that a linear homogeneous differential equation of the order n with constant real coefficients is given. It is known that $x^{50}\sin^4(3x)$ is one of the solutions of this equation. Find the smallest possible value of n.

Answer: 255

Solution:

This particular solution can be converted to the form

$$\frac{1}{8}x^{50}\cos 12x - \frac{1}{2}x^{50}\cos 6x + \frac{3}{8}x^{50}.$$

In order for a linear homogeneous differential equation to have such a quasi-polynomial as a solution, the roots of its characteristic equation must be the following numbers: $\lambda_{1,2} = \pm 12i$ are roots of multiplicity 51; $\lambda_{3,4} = \pm 6i$ roots of multiplicity 51; $\lambda_5 = 0$ roots of multiplicity 51.

We obtain that the characteristic equation must have at least 51·5 roots (taking into account multiplicity), so the minimum possible value of n is 255.

***Problem 10* (10 points)**

Players *A* and *B* play a chess match between themselves. Player *A* wins the game with a probability of 0.6. To even the odds, they agreed that player *A* wins if he wins three games, and *B* wins if he wins two games (draws are not counted). What is the probability of each player winning the match?

Answer: The probability of winning player *A* is equal to 0,4752. The probability of winning player *B* is equal to 0,5248.

Solution:

In any case, the match ends after four successful games. Let's imagine each outcome as a vector consisting of zeros and ones corresponding to the result of the games in order: 1 – «the first player wins the game», 0 – the second player wins the game. There are 16 possible outcomes in total, taking into account the possible early end of the match – 10. Using Bernoulli's formula, we have:

$$P(A) = C_4^4(0{,}6)^4 + C_4^3(0{,}6)^3(0{,}4) = (0{,}6)^4 + 4(0{,}6)^3(0{,}4) = 0{,}4752$$

is the probability of the 1st player winning the match;

$$P(B) = 1 - P(A) = 0{,}5248$$

is the probability of the 2nd player winning the match.
