ПЯТАЯ МЕЖДУНАРОДНАЯ МАТЕМАТИЧЕСКАЯ ОЛИМПИАДА
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Problem statements and solutions

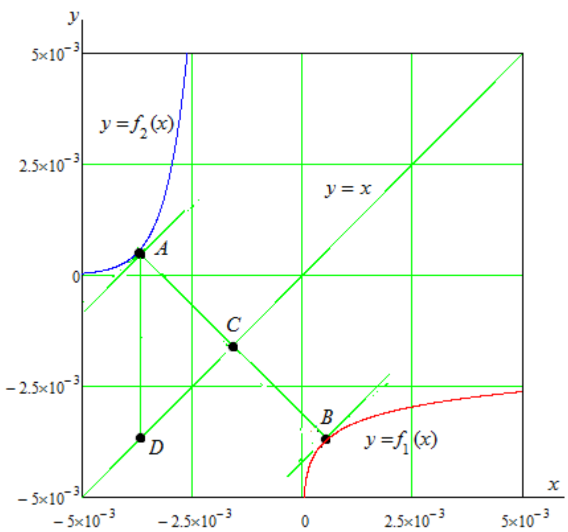
Problem 1 *(10 points)*

Find the distance between the graphs of the functions

$$f_1(x) = \frac{\ln x}{2025} \quad \text{and} \quad f_2(x) = e^{2025x}.$$

Answer: $\sqrt{2}(1 + \ln 2025)/2025$

Solution: : We note that the given functions are inverses of each other, and their graphs are symmetric with respect to the line y=x. Therefore, the desired distance



$$AB = 2AC = \sqrt{2}\,AD.$$

The coordinates of point A are determined from the condition

$$f'_2(x_A) = 1,$$

or equivalently,

$$2025 \cdot e^{2025 x_A} = 1$$

$$\Longrightarrow x_A = -\frac{\ln 2025}{2025};$$

the distance AD is found as the difference

$$AD = f_2(x_A) - x_A = \frac{1 + \ln 2025}{2025}.$$

Finally,

$$AB = \sqrt{2}\,AD = \sqrt{2}\,\frac{1+\ln 2025}{2025}.\blacksquare$$

Problem 2 ***(10 points)***

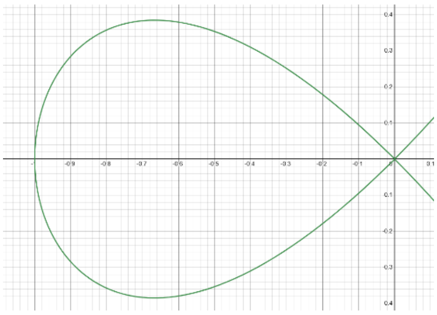
Calculate the area of the figure bounded by the curve loop $x^3 + x^2 - y^2 = 0$.

Answer: 8/15.

Solution:

By analyzing the graph of the curve, we conclude that it is symmetric with respect to the x-axis and forms a closed curve on the interval $-1 \le x \le 0,\ \ y(-1) = y(0) = 0$. Therefore, the loop is located on the interval [−1;0] and is bounded by the lines

$$y = \pm\sqrt{x^2(x+1)}.$$



The area enclosed by the loop is given by:

$$S = 2\int_{-1}^{0} |x|\sqrt{x+1}\,dx = \left\langle \begin{matrix} t = \sqrt{x+1},\ |x| = 1 - t^2,\ dx = 2tdt, \\ x = -1 \ \Rightarrow t = 0,\ \ x = 0 \ \Rightarrow t = 1 \end{matrix} \right\rangle =$$

$$= 2\int_{0}^{1} (1-t^2)t \cdot 2tdt = \cdots = \frac{8}{15}.\blacksquare$$

Problem 3 ***(10 points)***

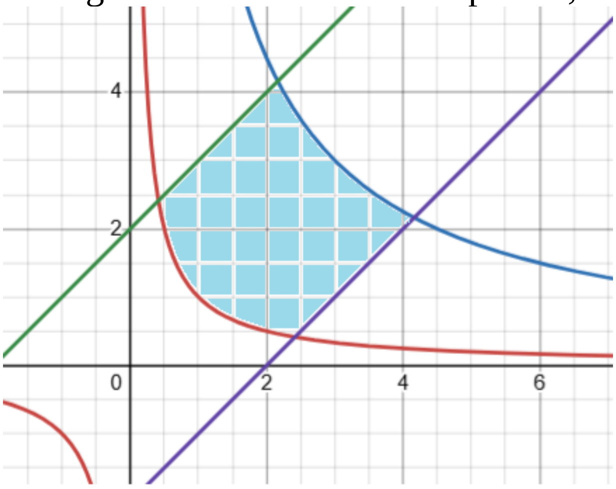
Calculate the double integral

$$\iint_D (x+y)dxdy,$$

where the region D is bounded by curves $xy = 1$, $xy = 9$, $y - x = 2$, $x - y = 2$ $(x > 0, y > 0)$.

Answer: 32.

Solution: The region of integration is a curvilinear trapezoid, as depicted in the figure.



Direct computation of the integral by reducing it to three iterated integrals is challenging. To simplify the problem, we perform the following change of variables:

$$xy = u,\ y - x = v.$$

a)

Express x and y in terms of *u* and *v*

$$x = \frac{1}{2}\left(-v + \sqrt{v^2 + 4u}\right), y = \frac{1}{2}\left(v + \sqrt{v^2 + 4u}\right).$$

Compute the Jacobian of the transformation

$$J(u,v) = \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{v^2+4u}} & \frac{1}{\sqrt{v^2+4u}} \\ \frac{v - \sqrt{v^2+4u}}{2\sqrt{v^2+4u}} & \frac{v + \sqrt{v^2+4u}}{2\sqrt{v^2+4u}} \end{vmatrix} = \frac{1}{\sqrt{v^2+4u}}.$$

Transform the integrand

$$f(x,y) = x + y \quad \Longrightarrow \quad f(x(u,v),(u,v)) = \sqrt{v^2+4u}$$

Describe the transformed region

$$D' = \begin{cases} 1 \le u \le 9, \\ -2 \le v \le 2. \end{cases}$$

Compute the integral using the change of variables formula

$$\iint\limits_D (x+y)dxdy = \iint\limits_{D\prime} \sqrt{v^2+4u} \cdot \frac{1}{\sqrt{v^2+4u}} dudv = \int_1^9 \left(\int_{-2}^{2} dv \right) du = 32.$$

b)

Compute the Jacobians of the forward and inverse transformations

$$J(x,y) = \begin{vmatrix} u'_x & v'_x \\ u'_y & v'_y \end{vmatrix} = \begin{vmatrix} y & -1 \\ x & 1 \end{vmatrix} = x + y; \quad J_1 = \frac{1}{J(x,y)} = \frac{1}{x+y}.$$

The region of integration in the new coordinate system *Ouv* is described as follows:

$$D' = \{(u,v) : 1 \le u \le 9, -2 \le v \le 2\}$$

Then, using the change of variables formula for a double integral, we find:

$$\iint_D (x+y)dxdy = \iint_{D'} (x+y)\cdot |J_1| dudv = \iint_{D'} dudv = \int_1^9 \left(\int_{-2}^{2} dv \right) du = 32. \blacksquare$$

Problem 4 ***(8 points)***

Find $x, y \in R$ that satisfy the system of equations

$$\begin{cases} 34x^2 - 22xy + 5y^2 = 98, \\ 16x^2 + 2xy - 3y^2 = 0. \end{cases}$$

Answer: $(1,-2)$, $(3,8)$, $(-3,-8)$, $(-1,2)$.

Solution:

By adding the equations, we obtain:

$$50x^2 - 20xy + 2y^2 = 98 \Longrightarrow (5x-y)^2 = 7^2.$$

By subtracting the second equation from the first, we find:

$$18x^2 - 24xy + 8y^2 = 98 \Longrightarrow (3x-2y)^2 = 7^2.$$

Thus, the original system can be rewritten as:

$$\begin{cases} (5x-y)^2 = 7^2, \\ (3x-2y)^2 = 7^2. \end{cases}$$

From this, we have four possible cases:

$$\begin{cases} 5x-y=7, \\ 3x-2y=7, \end{cases} \quad \begin{cases} 5x-y=7, \\ 3x-2y=-7, \end{cases} \quad \begin{cases} 5x-y=-7, \\ 3x-2y=7, \end{cases} \quad \begin{cases} 5x-y=-7, \\ 3x-2y=-7. \end{cases}$$

From these cases, we find four solutions: $(1,-2)$, $(3,8)$, $(-3,-8)$, $(-1,2)$. $\blacksquare$

Problem 5 ***(10 points)***

Calculate the limit of the sequence

$$\lim_{n\to\infty} n^2 \cdot \left(\sqrt[n]{n+a^2} - \sqrt[n]{n} \right).$$

Answer: a^2.

Solution:

a) By direct substitution, we obtain $[\infty\cdot(\infty^0-\infty^0)]$. Using the fundamental logarithmic identity, we transform the expression under the limit:

$$(n+a^2)^{\frac{1}{n}}-n^{\frac{1}{n}}=e^{\frac{ln(n+a^2)}{n}}-e^{\frac{ln\,n}{n}}=e^{\frac{ln\,n}{n}}\cdot\left(e^{\frac{ln(n+a^2)-ln\,n}{n}}-1\right)$$

$$=e^{\frac{ln\,n}{n}}\cdot\left(e^{\frac{ln\left(1+\frac{a^2}{n}\right)}{n}}-1\right).$$

Taking into account that $e^t-1\sim t$ and $ln(1+t)\sim t$ as $t\to 0$, we obtain:

$$\lim_{n\to\infty} n^2\cdot\left(\sqrt[n]{n+a^2}-\sqrt[n]{n}\right)=\lim_{n\to\infty}\frac{e^{\frac{ln\,n}{n}}\cdot\left(e^{\frac{ln\left(1+\frac{a^2}{n}\right)}{n}}-1\right)}{\frac{1}{n^2}}=$$

$$=\lim_{n\to\infty} e^{\frac{ln\,n}{n}}\cdot\lim_{n\to\infty}\frac{e^{\frac{ln\left(1+\frac{a^2}{n}\right)}{n}}-1}{\frac{1}{n^2}}=1\cdot\lim_{n\to\infty}\frac{e^{\frac{ln\left(1+\frac{a^2}{n}\right)}{n}}-1}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{\frac{ln\left(1+\frac{a^2}{n}\right)}{n}}{\frac{1}{n^2}}=$$

$$=\lim_{n\to\infty}\frac{ln\left(1+\frac{a^2}{n}\right)}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\frac{a^2}{n}}{\frac{1}{n}}=a^2.$$

b) A much faster approach is as follows:

$$\lim_{n\to\infty} n^2\cdot\left(\sqrt[n]{n+a^2}-\sqrt[n]{n}\right)=\lim_{n\to\infty} n^2\cdot\sqrt[n]{n}\cdot\left(\sqrt[n]{1+\frac{a^2}{n}}-1\right)=$$

$$=\lim_{n\to\infty} n^2\cdot\sqrt[n]{n}\cdot\left(1+\frac{a^2}{n^2}-1\right)=a^2.\blacksquare$$

Problem 6 ***(8 points)***

Find x at which $\Delta(x)=\begin{vmatrix}1 & 1 & 0 & 0 & 0\\ 2 & 1 & 1 & 0 & 0\\ 4 & 2 & 1 & 2 & 0\\ 8 & 4 & 2 & 2 & 1\\ x & -8 & -4 & -4 & -1\end{vmatrix}=0.$

Answer: $x=-15$.

Solution:
Let us transform the determinant:

$$\Delta(x) = 2 \cdot \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 \\ 8 & 4 & 2 & 1 & 1 \\ x & -8 & -4 & -2 & -1 \end{vmatrix}.$$

If we subtract the other four columns from the first column, we obtain:

$$\Delta(x) = 2 \cdot \begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 4 & 2 & 1 & 1 \\ x+15 & -8 & -4 & -2 & -1 \end{vmatrix}.$$

Next, we expand the determinant along the first column, yielding:

$$\Delta(x) = 2 \cdot (x+15) \cdot \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 4 & 2 & 1 & 1 \end{vmatrix}$$

The resulting fourth-order determinant is triangular, and therefore it equals the product of the diagonal elements, which is equal to 1.
Therefore $\Delta(x) = 2 \cdot (x+15) = 0$, from which we obtain $x = -15$. ■

Problem 7 ***(12 points)***
Find a general solution to the differential equation

$$x^2yy'' - 2x^2y'^2 + xyy' + y^2 = 0.$$

Answer: $y = \frac{C_2 x}{x^2 - C_1}$.

Solution:
We use the property of homogeneity of the equation with respect to y, y', y'' and transition to a new function $z(x)$: $y' = y \cdot z(x),\ \ y'' = y(z^2 + z'),$

$$x^2y^2(z^2+z') - 2x^2y^2z^2 + xy^2z + y^2 = 0 \Rightarrow x^2z' - x^2z^2 + xz + 1 = 0.$$

Let us note

$$(xz)' = z + xz' \Rightarrow x(xz)' - x^2z^2 + 1 = 0.$$

Let us make the substitution: $u(x) = xz$:

$$xu' - u^2 + 1 = 0 \Rightarrow \frac{du}{u^2-1} = \frac{dx}{x} \Rightarrow \frac{u-1}{u+1} = \frac{x^2}{C_1} \Rightarrow u = -\frac{x^2+C_1}{x^2-C_1}.$$

Returning to the function: $z(x)$:

$$z=\frac{u}{x}=-\frac{x^2+C_1}{x(x^2-C_1)}=\frac{1}{x}-\frac{2x}{x^2-C_1},$$

and then to: $y(x)$:

$$\frac{dy}{y}=z(x)dx=\left(\frac{1}{x}-\frac{2x}{x^2-C_1}\right)dx \Rightarrow y=\frac{C_2x}{x^2-C_1}. \blacksquare$$

Problem 8 ***(12 points)***

Find a general solution to the system of differential equations

$$\begin{cases}2zy'=y^2-z^2+1,\\ z'=z+y.\end{cases}$$

Answer: $y=\frac{C_1}{2}(x+C_2)-\frac{1}{C_1}-\frac{C_1}{4}(x+C_2)^2,\ z=\frac{1}{C_1}+\frac{C_1}{4}(x+C_2)^2.$

Solution:

From the second equation, we express $y=z'-z$ and substitute it into the first equation. We obtain:

$$2z\cdot z''-z'^2-1=0.$$

The order of the equation can be reduced by introducing a new independent variable z, and letting the unknown function be $z'=p(z)$, then

$$z''=p\frac{dp}{dz},$$

and the equation takes the form:

$$2zp\frac{dp}{dz}-p^2-1=0.$$

Separating variables, we get:

$$\frac{2pdp}{p^2+1}=\frac{dz}{z} \Rightarrow ln(p^2+1)=ln(z\cdot C_1) \Rightarrow p=\pm\sqrt{C_1z-1}.$$

Integrating both sides:

$$\frac{dz}{dx}=\pm\sqrt{C_1z-1} \Rightarrow \frac{dz}{\sqrt{C_1z-1}}=\pm dx \Rightarrow z=\frac{1}{C_1}+\frac{C_1}{4}(x+C_2)^2.$$

Returning to the y

$$y=z'-z=\frac{C_1}{2}(x+C_2)-\frac{1}{C_1}-\frac{C_1}{4}(x+C_2)^2.$$

Problem 9 ***(9 points)***

Calculate $y^{(21)}(0)$ for the function

$$y = e^{2x}\sin 2x.$$

Answer: $y^{(21)}(0) = -2^{31}$.

Solution:

Let us note, $y = Im\, e^{2x} \cdot e^{i\,2x} = Im\, e^{2x(1+i)}$.

Next, it is not difficult to find the derivative

$$\left[e^{2x(1+i)}\right]^{(21)} = 2^{21}(1+i)^{21}e^{2x(1+i)} = 2^{21}\left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{21} e^{2x(1+i)} =$$

$$= 2^{21}\left(\sqrt{2}\right)^{21}\left(\cos\frac{21\pi}{4} + i\sin\frac{21\pi}{4}\right)e^{2x(1+i)} = -2^{31}(1+i)e^{2x(1+i)}.$$

Finally,

$$y^{(21)}(0) = Im\left[e^{2x(1+i)}\right]^{(21)}\Big|_{x=0} = -Im\, 2^{31}(1+i)e^{2x(1+i)}\Big|_{x=0} = -2^{31}. \blacksquare$$

Problem 10 ***(11 points)***

How many times do you need to throw three dice so that the probability that at least one of the throws will produce a combination with three identical numbers (for example, 1-1-1, 2-2-2, etc.) becomes greater than a given number p, where $0 < p < 1$? How many throws will be required if $p = 0.5$?

Answer: $n > \frac{ln(1-p)}{ln\left(\frac{35}{36}\right)}$; $\quad n|_{p\,=\,0.5} = 25.$

Solution:

Step 1: Determine the probability of rolling three identical digits in one throw.

Each die has 6 faces, so the total number of possible outcomes when rolling three dice is $6 \cdot 6 \cdot 6 = 6^3 = 216$.

Out of these 216 outcomes, there are exactly 6 cases where all three dice show the same number (e.g., 1–1–1, 2–2–2, ..., 6–6–6). Therefore, the probability of rolling three identical digits in one throw is:

$$\text{P(all three identical)} = \frac{6}{216} = \frac{1}{36}.$$

Consequently, the probability that the rolled digits are not all identical is:

$$\text{P(not all three identical)} = 1 - \text{P(all three identical)} = \frac{35}{36}$$

Step 2: Find the probability of not rolling any triplets of identical digits in n throws. If we roll the three dice n times, the event "no triplet of identical digits appears" means that on each of the n rolls, the digits are either different or not all identical. Since each roll is independent, the probability of this event is:

$$\text{P(no triplets in n throws)} = \left(\frac{35}{36}\right)^n$$

Step 3: Compute the probability of rolling at least one triplet of identical digits.
The probability of rolling at least one triplet of identical digits is the complement of the previous probability:

$$\text{P(at least one triplet in n throws)} = 1 - \left(\frac{35}{36}\right)^n.$$

Step 4: Find the minimum *n* such that the probability exceeds a given $p \in (0,1)$:

$$1 - \left(\frac{35}{36}\right)^n > p \quad \Rightarrow \quad \left(\frac{35}{36}\right)^n < 1 - p \quad \Rightarrow \quad n\, ln\left(\frac{35}{36}\right) < ln(1-p) \quad \Rightarrow$$

$$\Rightarrow \quad n > \frac{ln(1-p)}{ln\left(\frac{35}{36}\right)}.$$

Thus, the answer is the smallest integer *n* satisfying:

$$n > \frac{ln(1-p)}{ln\left(\frac{35}{36}\right)}$$

Example Calculation for $p = 0.5$:

$$n > \frac{ln(1-0.5)}{ln\left(\frac{35}{36}\right)} = \frac{ln\, 0.5}{ln\left(\frac{35}{36}\right)} = \frac{-0{,}69314718}{-0{,}02817088} = 24.6,$$

Therefore, 25 rolls are required for the probability of rolling at least one triplet of identical digits to exceed 0.5.
