THE THIRD INTERNATIONAL MATHEMATICAL OLYMPIAD
Blagoveshchensk – Russia, 25 March 2023

Problem 1 *(8 points)*

Find the solution of the system of equations

$$\begin{cases} x + \dfrac{3x - y}{x^2 + y^2} = 3, \\ y - \dfrac{x + 3y}{x^2 + y^2} = 0. \end{cases}$$

Problem 2 *(10 points)*

A tangent line is drawn to the graph of the function $y = -(x^2/12) + x - 16/3$. This line intersects the graph of the function $y = 3|x + 6| - 7/3$ at points A and B. Find the radius of a circle circumscribed around a triangle with vertices at points A, B and $C(-6;\ -7/3)$, if $\angle CAB = 2arccos\left(3/\sqrt{10}\right) + \angle CBA$.

Problem 3 *(9 points)*

The matrices $A_{3\times2}$ and $B_{2\times3}$ are such that

$$AB = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}.$$

Find the matrix BA.

Problem 4 *(11 points)*

Find the general solution or the general integral of the equation

$$\frac{dy}{dx} = \frac{xy}{y^3 + x^2y + x^2}.$$

Problem 5 ***(10 points)***

Find the limit of the sum

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right).$$

Problem 6 ***(11 points)***

The sequence $\{u_n\}$ is given recurrently:

$$u_0 = 1,\; u_1 = 2,\; u_2 = 4, \qquad u_{n+1} = u_{n-2} + 2u_{n-1} + u_n, \qquad n \geq 2.$$

Find the sum of the series

$$\sum_{n=0}^{+\infty}\frac{u_n}{10^n}.$$

Problem 7 ***(9 points)***

Four players take turns tossing a coin, which each time with the same probability of 0.5 drops out «heads» or «tails».

The winner is the player who gets the «heads» for the first time.

Determine the probability of winning each of the players.

Problem 8 ***(13 points)***

Calculate the definite integral

$$I = \int_0^{2\pi}\frac{dx}{(a+b\cos^2 x)^2} \qquad (a>0, b>0).$$

Problem 9 ***(10 points)***

Find the general solution of the system of equations

$$\begin{cases}\ddot{x} - 2\dot{x} - 2x - \ddot{y} + 4\dot{y} = 0,\\ 2\dot{x} + x - \ddot{y} - 4\dot{y} + 2y = 0.\end{cases}$$

Problem 10 ***(9 points)***

Calculate the curvilinear integral

$$\oint_\Gamma \frac{x\,dy + y\,dx}{x^2+y^2}, \quad \Gamma = \{(x,y)\colon (x-1)^2 + (y-1)^2 = 1\}.$$